

On modules over Dedekind rings

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1. A ring in this paper always signifies an integral domain, and will be denoted by R . M will denote a unitary R -module.

In section 2 some properties of abelian groups will be generalized to R -modules. In most cases, R will be taken to be a Dedekind ring. The results of section 2 will be utilized in section 3 to obtain information about A -high submodules of M , A a submodule of M . In section 4, the results of section 2 will be employed in determining the structure of the tensor product of R -modules for several types of modules over a Dedekind ring.

2. Definition 1. M is said to be a *divisible* R -module, if $rM = M$ for all $0 \neq r \in R$.

Definition 2. Let P be a prime ideal in R . M is said to be *P -divisible*, if $PM = M$.

Definition 3. Let N be a submodule of M . N is said to be a *pure submodule* of M if for all $r \in R$ and for all $m \in M$, if $rm \in N$, then there exists an $n \in N$ such that $rm = rn$.

Definition 4. Let N be a submodule of M . N is said to be an *ideal pure submodule* of M if for every ideal I in R , $N \cap IM = IN$.

Ideal purity clearly implies purity.

Notation. Let $m \in M$, $\text{ord}(m) = \{r \in R \mid rm = 0\}$.

Definition 5. Let P be a prime ideal in R . M is said to be a *P -primary module* if for every $m \in M$ there exists a positive integer $k(m)$ such that $P^{k(m)} \subseteq \text{ord}(m)$.

Definition 6. Let P be a prime ideal in R . A submodule N of M is said to be *P -pure* in M if $N \cap P^k M = P^k N$ for every positive integer k .

Lemma 1. Let R be a Dedekind ring, P a prime ideal in R , and M a P -primary R -module. Then for every prime ideal Q in R , $Q \neq P$, $QM = M$.

Proof. Q is a maximal ideal in R , hence $Q \not\subseteq P$. Therefore $QM = M$ by [2, Lemma 4].

Lemma 2. Let R be a Dedekind ring, P a prime ideal in R , M a P -primary R -module, and N a submodule of M . If N is P -pure in M then N is ideal pure in M .

Proof. Let I be an ideal in R . Then $I = \prod Q^{k(Q)}$, Q running over the set of prime ideals in R , $k(Q)$ being a non-negative integer, $k(Q) = 0$ for all but finitely many Q [8, p. 274].

By Lemma 1, $IM = P^{k(P)}M$, and $IN = P^{k(P)}N$. Therefore, $N \cap IM = N \cap P^{k(P)}M = P^{k(P)}N = IN$.

Definition 7. An exact sequence $0 \rightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \rightarrow 0$ of R -modules is said to be (ideal) pure exact if $\text{im } \varphi$ is an (ideal) pure submodule of M .

Lemma 3. Let

$$(*) \quad 0 \rightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \rightarrow 0$$

be an exact sequence of R -modules:

a) If $(*)$ is pure exact then the sequence

$$0 \rightarrow L/rL \xrightarrow{\bar{\varphi}} M/rM \xrightarrow{\bar{\psi}} N/rN \rightarrow 0$$

is exact for every $r \in R$. $\bar{\varphi}$ and $\bar{\psi}$ are defined in the natural way.

b) If $(*)$ is ideal pure exact, then the sequence

$$0 \rightarrow L/IL \xrightarrow{\bar{\varphi}} M/IM \xrightarrow{\bar{\psi}} N/IN \rightarrow 0$$

is exact for every ideal I in R . $\bar{\varphi}$ and $\bar{\psi}$ are defined in the natural way.

Proof. Same as for abelian groups [3, Theorem 29.1].

The following are known facts concerning modules over a Dedekind ring:

Proposition 1. (STEINITZ [7]) Let R be a Dedekind ring and let M be a finitely generated R -module, then M is a direct sum of cyclic modules, and rank one torsion free modules.

Proposition 2. (KAPLANSKY [5, Theorem 1].) Let R be a Dedekind ring, and let M be a finitely generated R -module, then $M \cong M_t \oplus (M/M_t)$, M_t the torsion part of M .

Proposition 3. [5, p. 332] Let R be a Dedekind ring, and let M be a torsion module. Then M is a direct sum of P -primary modules.

Definition 8. Let S be a subset of M . S is said to be an *independent set* in M if for every positive integer k and for all $r_j \in R, m_j \in S$ ($1 \leq j \leq k$),

$$\sum_{j=1}^k r_j m_j = 0 \quad \text{implies} \quad r_j m_j = 0 \quad (1 \leq j \leq k).$$

Definition 9. Let S be a subset of M and let P be a prime ideal in R . S is said to be a *P -independent set* in M if for all positive integers k, l , and for all $r_j \in R, m_j \in S$ ($1 \leq j \leq k$),

$$\sum_{j=1}^k r_j m_j \in P^l M \quad \text{implies} \quad r_j \in P^l \quad (1 \leq j \leq k).$$

Lemma 4. Let R be a Dedekind ring, P a prime ideal in R , and S a P -independent set in M . Then S is independent.

Proof. Let $r_j \in R, m_j \in S$ ($1 \leq j \leq k$) for k a positive integer. Suppose $\sum_{j=1}^k r_j m_j = 0$. Then $\sum_{j=1}^k r_j m_j \in P^e M$ for every positive integer e . S is P -independent, hence $r_j \in P^e$ for every positive integer e ($1 \leq j \leq k$). However, R is Noetherian, so that $r_j = 0$ ($1 \leq j \leq k$) and S is therefore independent.

Lemma 5. Let P be a prime ideal in R , and let S be a P -independent subset of M . $\langle S \rangle$, the submodule of M generated by S , is P -pure in M .

Proof. Let $x \in \langle S \rangle \cap P^e M$, e a positive integer. Then $x = \sum_{j=1}^k r_j m_j, r_j \in R, m_j \in S$ ($1 \leq j \leq k$) and $x \in P^e M$. S is P -independent, so that $r_j \in P^e$ ($1 \leq j \leq k$). Therefore $x \in P^e \langle S \rangle$.

It has been observed [5, p. 332] that if R is a Dedekind ring, P a prime ideal in R , and M a P -primary R -module, then M may be viewed as an R_P -module (R_P the localization of R at the prime P). This may be done in the following manner.

Let $r/s \in R_P, r \in R, s \in R - P$, and let $m \in M, s \notin P$, hence there exists an $m' \in M$ such that $m = sm'$. Define $(r/s)m = rm'$. It is easily verified that this action of R_P on M gives M the structure of an R_P -module.

Lemma 6. Let R be a Dedekind ring, P a prime ideal in $R, S \subseteq M$. S is a (maximal) P -independent set in M iff S is a (maximal) PR_P -independent set in M .

Proof. 1) Suppose S is P -independent. Let $r_j \in R, s_j \in R - P, m_j \in S$ ($1 \leq j \leq k$), k a positive integer, and suppose that $x = \sum_{j=1}^k (r_j/s_j)m_j \in (PR_P)^e M, e$ a positive integer. $(PR_P)^e$ is a principal ideal: $(PR_P)^e = \langle r/s \rangle, r \in P^e, s \in R - P$. Therefore $x = (r/s)m$,

$m \in M$, and

$$\left(s \cdot \prod_{i=1}^k s_i\right) x = \sum_{j=1}^k \left(s \cdot \prod_{\substack{i=1 \\ i \neq j}}^k s_i\right) r_j m_j = \left(r \cdot \prod_{i=1}^k s_i\right) m \in P^e M.$$

S is P -independent, hence $\left(s \cdot \prod_{\substack{i=1 \\ i \neq j}}^k s_i\right) r_j \in P^e$ ($1 \leq j \leq k$). However, $s, s_i \notin P$ ($1 \leq i \leq k$) so that $r_j \in P^e$, and hence $r_j/s_j \in (PR_p)^e$ ($1 \leq j \leq k$).

Suppose that S is a maximal P -independent set in M . Let $0 \neq m \in M$. Then there exist $r_j \in R$ ($0 \leq j \leq k$) and $m_j \in S$ ($1 \leq j \leq k$), such that $r_0 m \neq 0$, $r_j m_j \neq 0$ ($1 \leq j \leq k$),

$$r_0 m + \sum_{j=1}^k r_j m_j \in P^e M, \quad e \text{ a positive integer, but } r_0 \notin P^e,$$

$$r_0 m + \sum_{j=1}^k r_j m_j \in (PR_p)^e M \quad (\text{we are here identifying } (r/1) \in R_p \text{ with } r \in R).$$

Suppose $r_0 \in (PR_p)^e$. Then $r_0 = r/s$, $r \in P^e$, $s \in R - P$. Then $r_0 s \in P^e$. However $s \notin P$ and hence $r_0 \in P^e$: a contradiction.

2) Suppose that S is a PR_p -independent subset of M . Let $r_j \in R$, $m_j \in S$ ($1 \leq j \leq k$), and suppose that $x = \sum_{j=1}^k r_j m_j \in P^e M$, e a positive integer. Then $x \in (PR_p)^e M$, and hence $r_j \in (PR_p)^e$ ($1 \leq j \leq k$). As was the case with r_0 above, this implies that $r_j \in P^e$ ($1 \leq k \leq j$).

Suppose that S is a maximal PR_p -independent set in M . Let $0 \neq m \in M$. Then there exist $r_j \in R$, $s_j \in R - P$ ($0 \leq j \leq k$), and $m_j \in S$, ($1 \leq j \leq k$) such that $(r_0/s_0)m \neq 0$, $(r_j/s_j)m_j \neq 0$ ($1 \leq j \leq k$).

$$x = (r_0/s_0)m + \sum_{j=1}^k (r_j/s_j)m_j \in (PR_p)^e M, \quad e \text{ a positive integer, but } r_0/s_0 \notin (PR_p)^e.$$

This implies $r_0 \notin P^e$.

$$x = (r/s)m', \quad r \in P^e, \quad s \in R - P, \quad m' \in M.$$

Therefore, $\left(s \cdot \prod_{i=1}^k s_i\right) x \in P^e$, but $\left(s \cdot \prod_{i=1}^k s_i\right) r_0 \notin P^e$. This implies that S is maximal P -independent in M .

Lemma 7. Let R be a Dedekind ring, P a prime ideal in R , and M a P -primary R -module. M is P -divisible iff M is PR_p divisible.

Proof. 1) Suppose that M is P -divisible. Obviously, $P \subseteq PR_p$. Hence, $M = PM \subseteq PR_p M \subseteq M$, and M is PR_p -divisible.

2) Suppose that M is PR_p -divisible. Let $m \in M$. There exist $p \in P$, $s \in R - P$, and $m' \in M$, such that $(p/s)m' = m$. Hence $sm \in PM$, but $s \notin P$. This implies that $m \in PM$.

Theorem 1. *Let R be a Dedekind ring, P a prime ideal in R , and M a P -primary R -module. Let S be a maximal P -independent set in M . Then $M/\langle S \rangle$ is P -divisible.*

Proof. Let $0 \neq m \in M$. By Lemma 6, there exist $r_j \in R, s_j \in R - P$ ($0 \leq j \leq k$) and $m_j \in S$ ($1 \leq j \leq k$) such that

$$x = (r_0/s_0)m + \sum_{j=1}^k (r_j/s_j)m_j \in (PR_p)^e M,$$

e a positive integer, $(r_0/s_0)m \neq 0$, and $(r_j/s_j)m_j \neq 0$ ($1 \leq j \leq k$), $(r_0/s_0) \notin (PR_p)^e$, $(r_0/s_0) \in (PR_p)^{e'}$ for $0 \leq e' < e$. This implies that $(r_j/s_j) \in (PR_p)^{e'}$ ($1 \leq j \leq k$). $(PR_p)^{e'}$ is a principal ideal in R_p ; hence $(PR_p)^{e'} = (r/s)$, $r \in P^{e'}$, $s \in R - P$. Therefore $x = (r/s)m'$, $m' \in M$, and $(r_j/s_j) = (r/s)(r'_j/s'_j)$, $r'_j \in R, s'_j \in R - P$ ($0 \leq j \leq k$), $r_0 \in R - P$.

This yields that $(r/s)t = 0$, where

$$t = (r'_0/s'_0)m - m' + \sum_{j=1}^k (r'_j/s'_j)m_j = 0.$$

Put $\bar{y} = y + \langle S \rangle$ for $y \in M$. Clearly, $\bar{y} \in PR_p(M/\langle S \rangle)$ holds for every $y \in M$ for which $(PR_p)^0 \subseteq \text{ord}(y)$.

Suppose that $\bar{y} \in PR_p(M/\langle S \rangle)$ for every $y \in M$ for which $(PR_p)^{k'} \subseteq \text{ord}(y)$ ($0 \leq k' < e$). Then $\bar{t} \in PR_p(M/\langle S \rangle)$. This implies that $(r'_0/s'_0)m - m' \in PR_p(M/\langle S \rangle)$. However, $m' \in (PR_p)^{e-e'}M \subseteq (PR_p)M$, so that $(r'_0/s'_0)\bar{m} \in (PR_p)(M/\langle S \rangle)$. $(r'_0/s'_0) \notin PR_p$, so that $\bar{m} \in (PR_p)(M/\langle S \rangle)$. $M/\langle S \rangle$ is therefore PR_p -divisible, and hence P -divisible by Lemma 7.

Lemma 8. *Let R be a ring for which every finitely generated ideal is principal. Let I be an ideal in R , and let A and B be R -modules. Then $I(A+B) = IA + IB$.*

Proof. Clearly $I(A+B) \subseteq IA + IB$. Let $x \in IA + IB$. Then

$$x = \sum_{j=1}^k (i_j a_j + i'_j b_j), \quad i_j, i'_j \in I, \quad a_j \in A, \quad b_j \in B \quad (1 \leq j \leq k).$$

The ideal $\langle i_j, i'_j \mid 1 \leq j \leq k \rangle = \langle i \rangle$, $i \in I$. Therefore $i_j = r_j i$, $i'_j = r'_j i$; $r_j, r'_j \in R$ ($1 \leq j \leq k$). Hence $x = i \left(\sum_{j=1}^k r_j a_j + \sum_{j=1}^k r'_j b_j \right) \in I(A+B)$.

3. Definition 10. Let A be a submodule of M . A submodule B of M is said to be A -high if $A \cap B = 0$, and if for every submodule C of M , $B \subsetneq C$, implies that $A \cap C \neq 0$.

Lemma 9. *Let A be a submodule of M , B an A -high submodule of M , and $N = A \oplus B$. Then M/N is a torsion module.*

Proof. Let $m \in M$, $m \notin N$. Then there exists a non-zero $a \in A \cap \langle B, m \rangle$. Let $a = b + rm$, $b \in B$, $r \in R$. If $r = 0$, then $a \in B$, contradicting the fact that $A \cap B = 0$. Therefore $r\bar{m} = \bar{0} = N$.

Lemma 10. *Let R be a Dedekind ring, and let M, A, B , and N be as in Lemma 9. Let P be a prime ideal in R , and let $m \in M$. If $Pm \subseteq B$, then $m \in N$.*

Proof. If $m \in B$, then $m \in N$. Suppose $m \notin B$. Then there exists a non-zero $a \in A \cap \langle B, m \rangle$, $a = b + rm$, $b \in B$, $r \in R$. Since $Pm \subseteq B$, and $A \cap B = 0$, we have that $r \notin P$. However, P is a maximal ideal in R , so that there exist $p \in P$, and $u, v \in R$, such that $ur + vp = 1$. $m = rum + vpm = u(a - b) + vpm \in N$.

Theorem 2. *Let R be a principal ideal ring, and let M, A, B , and N be as above. Let π_A be the projection of N onto A . $M = N$ iff for every $m \in M$, and for every prime ideal P in R , $Pm \subseteq N$ implies that $\pi_A(Pm) \subseteq PA$.*

Proof. 1) Suppose that for every $m \in M$, and for every prime ideal P in R , $Pm \subseteq N$ implies that $\pi_A(Pm) \subseteq PA$. Let $m \in M$, and suppose that $Pm \subseteq N$, P a proper prime ideal in R . Then $Pm \subseteq PA \oplus B$. $P = \langle p \rangle$, $p \in P$, hence there exist $a \in A$, and $b \in B$ such that $pm = pa + b$, or $p(m - a) = b$. By Lemma 10, $m - a \in N$, and hence $m \in N$.

We have shown that for every $m \in M$, $m \notin N$, $P \subseteq \text{ord}(\bar{m})$ for every prime ideal P in R . By Lemma 9, M/N is a torsion module. A contradiction.

2) Suppose $M = N = A \oplus B$, and let P be a prime ideal in R . By Lemma 8, $PM = PA \oplus PB$, and hence $\pi_A(PM) = PA$.

Corollary. *Let R be a Dedekind ring, and let M, A, B and N be as above. If M is a torsion module, then the statement of Theorem 2 remains true.*

Proof. By Proposition 3 we may consider M to be a P -primary module. M is then an R_p module, R_p a principal ideal ring. We may therefore employ Theorem 2.

Notation: Let I be an ideal in R . Then $M[I] = \{m \in M \mid I \subseteq \text{ord}(m)\}$.

Theorem 3. *Let R be a Dedekind ring, M, A, B , and N as above, and let P be a prime ideal in R . Then $(M/N)[P] \cong [\langle PM, B \rangle \cap A] / PA$.*

Proof. If R is a principal ideal ring, then the theorem may be proved as in the case of abelian groups [4]. In the general case $(M/N)[P]$ is a P -primary module, and hence an R_p -module, so that the theorem remains true.

Several results concerning abelian groups may be generalized to modules over a Dedekind ring R as a result of Theorem 3; see [4]. For example, KULIKOV's theorem stating that a bounded pure subgroup of an abelian group is a direct summand [3, Theorem 27.5] can thus be generalized. This result has already been obtained by KAPLANSKY [5, Theorem 5] in a different manner.

4. Notation. The tensor product \otimes_R will be denoted by \otimes .

Lemma 11. Let $\langle m \rangle$ be a cyclic R -module, N an arbitrary R -module. Then $\langle m \rangle \otimes N \cong N/\text{ord}(m) \cdot N$.

Proof. Same as for abelian groups [3, p. 255].

Theorem 4. Let R be a Dedekind ring, and let

$$(*) \quad 0 \rightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \rightarrow 0$$

be an ideal pure exact sequence of R -modules. Then for every R -module M , the sequence

$$(**) \quad 0 \rightarrow A \otimes \xrightarrow{\varphi \otimes 1_M} B \otimes M \xrightarrow{\psi \otimes 1_M} C \otimes M \rightarrow 0$$

is exact.

Proof. 1) Let M be a torsion module, and let M' be a finitely generated submodule of M . By Proposition 1, M' is a direct sum of cyclic modules. The sequence

$$(***) \quad 0 \rightarrow A \otimes M' \xrightarrow{\varphi \otimes 1_{M'}} B \otimes M' \xrightarrow{\psi \otimes 1_{M'}} C \otimes M' \rightarrow 0$$

is therefore exact by Lemma 11 and Lemma 3.

For every R -module L ,

$$L \otimes M \cong \varinjlim \{L \otimes M' \mid M' \text{ a finitely generated submodule of } M\}$$

so that $(**)$ is exact by [6, Theorem 2.13].

2) Let M be a torsion free module. Then M is flat [6, Theorem 4.23] so that $(**)$ is exact.

3) Let M be an arbitrary R -module, and let M' be a finitely generated submodule of M . Proposition 2 together with 1) and 2) yield that $(***)$ is exact. We may proceed as in 1) to obtain that $(**)$ is exact.

Lemma 12. Let R be a Dedekind ring, and let J be an injective R -module. Then for every R -module M , $M \otimes J \cong (M/M_t) \otimes J$.

Proof. The sequence

$$0 \rightarrow M_t \rightarrow M \rightarrow M/M_t \rightarrow 0$$

is exact, hence the sequence

$$M_t \otimes J \rightarrow M \otimes J \rightarrow (M/M_t) \otimes J \rightarrow 0$$

is exact. J is divisible so that $M_t \otimes J = 0$. Therefore, $M \otimes J \cong (M/M_t) \otimes J$.

Let S be a maximal independent subset of an R -module M , and let $S_0 = \{x \in S \mid x \text{ is torsion free}\}$. It is easy to verify that the cardinality of S_0 , $|S_0|$, is independent of the choice of S . We may therefore give the following

Definition 11. Let M , S , and S_0 be as above. Then $r_0(M) = |S_0|$ is called the *torsion free rank* of M .

Theorem 5. Let R be a Dedekind ring, let J be a torsion free injective R -module, and let M be an arbitrary R -module. Then $M \otimes J \cong \sum_{r_0(M)} \oplus J$.

Proof. By Lemma 12 we may assume that M is torsion free. Let S be a maximal independent subset of M . The sequence

$$0 \rightarrow \langle S \rangle \rightarrow M \rightarrow M/\langle S \rangle \rightarrow 0$$

is exact. J is flat so that the sequence

$$0 \rightarrow \langle S \rangle \otimes J \rightarrow M \otimes J \rightarrow (M/\langle S \rangle) \otimes J \rightarrow 0$$

is exact. $M/\langle S \rangle$ is a torsion module, and J is divisible. Hence $(M/\langle S \rangle) \otimes J = 0$, and $M \otimes J \cong \langle S \rangle \otimes J \cong \sum_{r_0(M)} \oplus J$.

Corollary. Let R be a Dedekind ring, K the quotient field of R , M and N torsion free R -modules. Then there exist embeddings,

$$\sum_{r_0(M)r_0(N)} R \rightarrow M \otimes N, \text{ and } M \otimes N \rightarrow \sum_{r_0(M)r_0(N)} K.$$

Proof. Let S be a maximal independent subset of M , and let T be a maximal independent subset of N . Then

$$\langle S \rangle \cong \sum_{r_0(M)} \oplus R, \text{ and } \langle T \rangle \cong \sum_{r_0(N)} \oplus R.$$

The sequence

$$0 \rightarrow \langle S \rangle \otimes T \rightarrow M \otimes N$$

is exact ([1] Theorem 3, and [2] Lemma 6), and

$$\langle S \rangle \otimes \langle T \rangle \cong \sum_{r_0(M)r_0(N)} \oplus R.$$

N is flat, hence there exists an exact sequence $0 \rightarrow N \xrightarrow{\varphi} N \otimes K$. However, M is also flat, so that the sequence

$$0 \rightarrow M \otimes N \xrightarrow{1_M \otimes \varphi} (M \otimes N) \otimes K$$

is exact. By Theorem 5

$$(M \otimes N) \otimes K \cong \sum_{r_0(M)r_0(N)} \oplus K.$$

Lemma 13. Let M be a P -primary module, and let N be a P -divisible module. Then $M \otimes N = 0$.

Proof. Let $m \in M$, $n \in N$, and let e be a positive integer such that $P^e \subseteq \text{ord}(m)$. Since $N = P^e N$, there exist $r_i \in P^e$, $n_i \in N$, $1 \leq i \leq k$ such that

$$n = \sum_{i=1}^k r_i n_i, \quad m \otimes n = \sum_{i=1}^k r_i m \otimes n_i = 0.$$

Theorem 6. Let R be a Dedekind ring, P a prime ideal in R , M and N P -primary R -modules, and S a maximal P -independent subset of M . Then $M \otimes N \cong \langle S \rangle \otimes N$.

Proof. The sequence

$$0 \rightarrow \langle S \rangle \rightarrow M \rightarrow M/\langle S \rangle \rightarrow 0$$

is ideal pure exact by Lemmata 5 and 2. By Theorem 4, the sequence

$$0 \rightarrow \langle S \rangle \otimes N \rightarrow M \otimes N \rightarrow (M/\langle S \rangle) \otimes N \rightarrow 0$$

is exact. By Theorem 1, $M/\langle S \rangle$ is P -divisible. Hence by Lemma 13, $(M/\langle S \rangle) \otimes N = 0$, and $M \otimes N \cong \langle S \rangle \otimes N$.

Corollary. Let R be a Dedekind ring, and let M and N be torsion R -modules. Then $M \otimes N$ is a direct sum of cyclic modules.

Proof. By Proposition 3, we may assume that M and N are P -primary modules. Let S be a maximal P -independent subset of M , and let T be a maximal P -independent subset of N . By Theorem 6, $M \otimes N \cong \langle S \rangle \otimes \langle T \rangle$. Proposition 1 and Lemma 11 yield that $S \otimes T$ is a direct sum of cyclic modules.

References

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